On Pólya frequency functions. II: Variation-diminishing integral operators of the convolution type.

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1. Introduction and statement of results[1]).

A real matrix $A = \|a_{ik}\|$ $(i = 1, \ldots, m;\ k = 1, \ldots, n)$ is said to be *totally positive* if all its minors, of any order, are non-negative. In 1930[2]) the author showed that if A is totally positive, then the linear transformation

$$(1)\qquad y_i = \sum_{k=1}^{n} a_{ik} x_k \qquad (i = 1, \ldots, m)$$

is variation-diminishing in the sense that if $v(x_k)$ denotes the number of variations of sign in the sequence $\{x_k\}$ and $v(y_i)$ the corresponding number in the sequence $\{y_i\}$, then we always have the inequality $v(y_i) \leq v(x_k)$. In the same paper of 1930 the author showed that (1) is certainly variation-diminishing if the matrix A does not possess two minors of equal orders and of opposite signs; also the converse holds to a certain extent: If (1) is variation-diminishing, then A cannot have two minors of equal orders and of opposite signs, *provided the rank of A is* $= n$. The necessary and sufficient conditions in order that (1) be variation-diminishing were found in 1933 by TH. MOTZKIN[3]). Since they will be used in this paper we state them here as follows: Let r be the rank of A; then A should not have two minors of equal orders and of opposite signs if their common order is $< r$, while if their common order is $= r$ then again they should never be of opposite signs if they belong to the same combination of r columns of A.

A function $\Lambda(x)$, $-\infty < x < \infty$, is called a Pólya frequency function (abbreviated P. f. f.) if it satisfies the following three characteristic conditions:

1. $\Lambda(x)$ is measurable.

1) A résumé of the results of this paper has appeared under the same title in the *Proceedings of the National Academy of Sciences*, **34** (1948), pp. 164—169.

2) I. J. SCHOENBERG, Über variationsvermindernde lineare Transformationen, *Math. Zeitschrift*, **32** (1930), pp. 321—328.

3) TH. MOTZKIN, *Beiträge zur Theorie der linearen Ungleichungen*, Doctoral dissertation, Basel, 1933 (Jerusalem, 1936), 69 pp., especially Chap. IV.

2. If $x_1 < x_2 < \ldots < x_n$ and $t_1 < t_2 < \ldots < t_n$ then $\det \|\Lambda(x_i - t_j)\|_{1,n} \geqq 0$, $n = 1, 2, 3, \ldots$. For $n = 1$ this means that $\Lambda(x) \geqq 0$.

3. Finally

$$0 < \int_{-\infty}^{\infty} \Lambda(x)\, dx < +\infty.$$

In our previous paper on Pólya frequency functions[4]) it was shown that a P. f. f. $\Lambda(x)$ has a bilateral Laplace transform of the form

$$\int_{-\infty}^{\infty} e^{-xs} \Lambda(x)\, dx = \frac{1}{\Psi(s)} \qquad (-c < Rs < c \text{ for some } c > 0), \tag{2}$$

where $\Psi(s)$ is an entire function of the type II of Laguerre, Pólya and Schur

$$\Psi(s) = c e^{-\gamma s^2 - \delta s} \prod_{\nu=1}^{\infty} (1 + \delta_\nu s)\, e^{-\delta_\nu s} \quad \left(c > 0,\ \gamma \geqq 0,\ 0 < \gamma + \sum_{1}^{\infty} \delta_\nu^2 < \infty\right), \tag{3}$$

and that conversely, the reciprocal of a function $\Psi(s)$ of this type allows of a representation (2) where $\Lambda(x)$ is a P. f. f.

All Pólya frequency functions $\Lambda(x)$ are everywhere continuous with the single exception of the so-called truncated exponential

$$\Lambda(x) = \begin{cases} e^{-x} & \text{if } x \geqq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

and all functions arising from it by a change of scale and origin. Further notable examples of P. f. f. are $\Lambda(x) = e^{-x^2}$, $e^{-|x|}$, $e^{x - e^x}$, $1/\cosh x$.

Let $f(x)$ be a real function defined for all real x. The number $v(f)$, of variations of sign of $f(x)$ in the range $(-\infty, \infty)$ is defined as follows: If S: $x_1 < x_2 < \ldots < x_n$ is an arbitrary finite increasing sequence of reals, then

$$v(f) = \sup_S v(f(x_i)) \qquad (0 \leqq v(f) \leqq \infty).$$

Let now $L(t)$ be a given real function of bounded variation in the range $-\infty < t < \infty$ which we normalize by the conditions that $L(-\infty) = 0$, $2L(t) = L(t+0) + L(t-0)$; we also rule out the trivial case when $L(t) \equiv 0$. Let us consider the integral transformation

$$g(x) = \int_{-\infty}^{\infty} f(x - t)\, dL(t), \tag{4}$$

where $f(x)$ is an arbitrary continuous and bounded function. We say that (4)

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[4]) I. J. Schoenberg, On totally positive functions, Laplace integrals and entire functions of the Laguerre—Pólya—Schur type, *Proceedings of the National Academy of Sciences*, **33** (1947), pp. 11—17. A detailed paper will appear under the title "On Pólya frequency functions. I: Totally positive functions and their Laplace transforms" probably in the *Transactions of the American Mathematical Society*.

is *variation-diminishing* if (4) always implies the inequality

(5) $$v(g) \leqq v(f).$$

Our main result is the following

Theorem 1. *The transformation* (4) *is variation-diminishing if and only if $L(t)$ is either, up to the sign, a cumulative Pólya frequency function*

(6) $$L(t) = \varepsilon \int_{-\infty}^{t} \Lambda(u)\, du,$$

where $\varepsilon = \pm 1$ and $\Lambda(x)$ is a Pólya frequency function, or else $L(t)$ is a step-function with only one jump.

On combining this theorem with (2) and (3) we may restate our result without a distinction of two cases as

Theorem 2. *The transformation* (4) *is variation-diminishing if and only if $L(t)$ has a bilateral Laplace-Stieltjes transform of the form*

(7) $$\int_{-\infty}^{\infty} e^{-st}\, dL(t) = C' e^{\gamma s^2 + \delta s} \prod_{\nu=1}^{\infty} \frac{e^{\delta_\nu s}}{1+\delta_\nu s} \qquad (-c < Rs < c, \text{ for some } c > 0),$$

where $C' \gtrless 0$, $\gamma \geqq 0$, δ, δ_ν real, $\sum_{1}^{\infty} \delta_\nu^2 < \infty$.

It should be noticed that the trivial case when $\gamma = \delta_\nu = 0$, which was excluded in (3), corresponds to a step-function $L(t)$ with only one jump, in which case our transformation (4) becomes

(8) $$g(x) = C' f(x+\delta)$$

and is evidently variation-diminishing.

2. Proof of the direct part of Theorem 1.

If (6) holds, the (4) becomes, assuming $\varepsilon = 1$,

(9) $$g(x) = \int_{-\infty}^{\infty} \Lambda(x-t) f(t)\, dt$$

and we are to show that (9) is variation-diminishing within the class of bounded continuous function $f(x)$. This we shall now do even for the wider class of *measurable* and *bounded* functions $f(x)$. Indeed if $v(f) = \infty$ then (5) is trivially verified. Hence we may assume $v(f) < \infty$, $v(f) = m$ say.

Let

(10) $$F(t) = \int_{0}^{t} f(t)\, dt \qquad (-\infty < t < \infty);$$

then (9) becomes

$$g(x)=\int_{-\infty}^{\infty} \Lambda(x-t)\,dF(t). \tag{11}$$

If $x_0<x_1<\ldots<x_n$ we are to show that

$$v(g(x_i))\leqq m, \tag{12}$$

and without losing generality we may assume that $v(g(x_i))=n$, that is

$$g(x_0), g(x_1), \ldots, g(x_n) \qquad \text{alternate in sign.} \tag{13}$$

Clearly (11) converges uniformly in the $n+1$ points x_i; hence we can find A such that the quantities

$$g_i=\int_{-A}^{A} \Lambda(x_i-t)\,dF(t) \qquad (i=0, 1, \ldots, n), \tag{14}$$

also alternate in sign, i. e. $v(g_i)=n$.

But (14) are ordinary Stieltjes integrals. We may therefore divide $(-A, A)$ into N equal parts by $t_0=-A<t_1<\ldots<t_N=A$ such that

$$\bar{g}_i=\sum_{\nu=1}^{N} \Lambda(x_i-t_\nu)\,(F(t_\nu)-F(t_{\nu-1})) \tag{15}$$

are all so close to the respective integrals (14) that we also have

$$v(\bar{g}_i)=n. \tag{16}$$

Since by assumption $v(f)=m$ we see, by (10), that the number of variations of sign in the sequence $F(t_\nu)-F(t_{\nu-1})$ $(\nu=1, \ldots, N)$ is $\leqq m$. But (15) is a variation-diminishing linear transformation in view of the total positivity of the matrix $\|\Lambda(x_i-t_\nu)\|$. By (16) we conclude that $n\leqq m$, hence (12) is established.

3. Proof of the converse part of Theorem I in case when $L(t)$ is continuously differentiable.

Writing $\lambda(t)=L'(t)$ which we assume continuous and clearly also in the Lebesgue class $L(-\infty, \infty)$, we now assume the transformation

$$g(x)=\int_{-\infty}^{\infty} \lambda(x-t)\,f(t)\,dt \tag{17}$$

to be variation-diminishing for bounded continuous $f(t)$ and we are to conclude that $\lambda(x)$ is, up to the sign, a Pólya frequency function. By obvious continuity arguments the variation-diminishing property of (17) immediately generalizes to the case when $f(t)$ is a step-function with a finite number of jumps.

Let

(18) $$x_1 < x_2 < \ldots < x_n;\; t_1 < t_2 < \ldots < t_n;$$

we want to show first that

(19) $$y_i' = \sum_{\nu=1}^{n} \lambda(x_i - t_\nu)\, y_\nu \qquad (i = 1, \ldots, n)$$

is a variation-diminishing linear transformation. To this end, given the y_ν, we define a step-function $f(t)$ as follows: $f(t) = \frac{1}{2h} y_\nu$ if $t_\nu - h \leqq t \leqq t_\nu + h$, $f(t) = 0$ outside the n intervals $[t_\nu - h, t_\nu + h]$ which are assumed not to overlap.

For the values of $g(x_i)$ as given by (17) we now find that

(20) $$g(x_i) = \sum_{\nu=1}^{n} \left(\frac{1}{2h} \int_{t_\nu - h}^{t_\nu + h} \lambda(x_i - t)\, dt \right) y_\nu \qquad (i = 1, \ldots, n).$$

Since $v(f) = v(y_\nu)$, while $v(g(x_i)) \leqq v(g)$, we have

(21) $$v(g(x_i)) \leqq v(y_\nu),$$

showing that (20) is variation-diminishing. Letting $h \to 0$, the continuity of $\lambda(x)$ implies $\lim\limits_{h \to 0} g(x_i) = y_i' = \sum\limits_{\nu=1}^{n} \lambda(x_i - t_\nu)\, y_\nu$, and, by (21), we have

$$v(y_i') \leqq \varliminf_{h \to 0} v(g(x_i)) \leqq v(y_\nu).$$

Thus (19) is indeed variation-diminishing.

This result implies in particular that $\lambda(x)$ never changes sign; for indeed, if it did, we could arrange to have $\lambda(x_1 - t_1)$ and $\lambda(x_2 - t_1)$ of opposite signs and this would contradict the inequality $v(y_i') \leqq v(y_\nu)$ for the values $y_1 = 1$, $y_2 = \ldots = y_n = 0$, if substituted into (19). Thus without loss of generality we may assume to have that

(22) $$\lambda(x) \geqq 0 \quad \text{for all } x, \text{ in particular } \quad \lambda(0) > 0,$$

since the last condition always obtains after a suitable shift of origin.

Let again $\{x_i\}$ *and* $\{t_j\}$ *satisfy* (18). *We wish to prove now that*

(23) $$D_n = \det \| \lambda(x_i - t_j) \|_{1,n} \geqq 0.$$

This is the main point of the proof. We shall establish it first for $n = 2$:

(24) *If* $x_1 < x_2$, $t_1 < t_2$, *then* $D_2 = \begin{vmatrix} \lambda(x_1 - t_1) & \lambda(x_1 - t_2) \\ \lambda(x_2 - t_1) & \lambda(x_2 - t_2) \end{vmatrix} \geqq 0.$

Indeed, suppose we had

(25) $$D_2 < 0.$$

Then let $x_3 = t_3 = \tau > \max(x_2, t_2)$ and consider

$$(26) \qquad D_3 = \begin{vmatrix} \lambda(x_1-t_1) & \lambda(x_1-t_2) & \lambda(x_1-\tau) \\ \lambda(x_2-t_1) & \lambda(x_2-t_2) & \lambda(x_2-\tau) \\ \lambda(\tau-t_1) & \lambda(\tau-t_2) & \lambda(0) \end{vmatrix}.$$

Since $\lambda(x) \geqq 0$ and the integral

$$\int_0^\infty (\lambda(x_1-\tau)+\lambda(x_2-\tau)+\lambda(\tau-t_1)+\lambda(\tau-t_2)\, d\tau$$

converges, we can certainly choose $\tau > \max(x_2, t_2)$ such that each of the four quantities under the integral sign are as small as we please. But then (25), (26) and $\lambda(0) > 0$ clearly imply that

$$D_3 = D_2\, \lambda(0) + \text{ (an arbitrarily small quantity) } < 0$$

for some appropriate value of τ. On the other hand, at least one of the four elements of D_2 is positive, $\lambda(x_1-t_2)$ say, and this implies that

$$(27) \qquad \begin{vmatrix} \lambda(x_1-t_2) & \lambda(x_1-\tau) \\ \lambda(\tau-t_2) & \lambda(0) \end{vmatrix} = \lambda(x_1-t_2)\, \lambda(0) + \text{(small quantity)} > 0.$$

But now we have a contradiction with the properties of variation-diminishing transformation stated in the first paragraph of our introduction: (26) is the determinant of a non-singular variation-diminishing transformation; as such it cannot have two minors, such as (25) and (27), which are of equal orders less than its rank 3, and of opposite signs.

We may now turn to a proof of the general inequality (23). Firstly we recall[5]) that the property (24) means that $\lambda(x)$ is logarithmically concave; this fact and the summability of $\lambda(x)$ imply that

$$(28) \qquad \lim_{x \to \pm\infty} \lambda(x) = 0,$$

even exponentially. Secondly, let $\xi_0, \xi_1, \ldots$ be an infinite sequence of real numbers having the following properties:

1. The sequence $\{\xi_\nu\}$ is monotone increasing.
2. The sequence $\{\xi_\nu\}$ contains every element of the finite sequences $\{x_i\}$ and $\{t_j\}$ appearing in (23).
3. For sufficiently large ν, $\{\xi_\nu\}$ is made up of consecutive integers.

N and r being positive integers let us consider the following three determinants

$$(29) \qquad \begin{aligned} D_{N+1} &= \det \|\lambda(\xi_i - \xi_j)\| & &(i, j = 0, 1, \ldots, N), \\ D_n^* &= \det \|\lambda(\xi_{r_i} - \xi_{r_j})\| & &(i, j = 0, 1, \ldots, n-1), \\ D_{n+1}^* &= \det \|\lambda(\xi_{r_i} - \xi_{r_j})\| & &(i, j = 0, 1, \ldots, n). \end{aligned}$$

From the property 3 of the sequence $\{\xi_\nu\}$, together with (28), it is clear that

[5]) See the paper mentioned in footnote [4]).

$\lambda(\xi_{ri}-\xi_{rj})\to 0$ as $r\to\infty$, provided that $i\neq j$. By (29) we have $D_n^*\to(\lambda(0))^n$ and $D_{n+1}^*\to(\lambda(0))^{n+1}$; thus we now see that

$$D_n^*>0,\ D_{n+1}^*>0, \tag{30}$$

provided r is chosen large enough. If we now choose $N>rn$, then clearly D_n^* and D_{n+1}^* are minors of D_{N+1}; by the property 2 we may moreover assume also D_n to be a minor of D_{N+1}, by further increasing N, if necessary. We may now prove (23) as follows: We know from the first result of this section that D_{N+1} is the determinant of a variation-diminishing transformation; by (30) the rank of D_{N+1} is $\geqq n+1$. By MOTZKIN's theorem of our introduction we conclude that the two minors D_n and D_n^* cannot have opposite signs, hence $D_n^*>0$ implies that $D_n\geqq 0$. But then $\lambda(x)$ is indeed a Pólya f. f.

4. General proof of the converse part of Theorem 1.

Let $f(t)$ be continuous and bounded and let $g(x)$ be defined by

$$g(x)=\int_{-\infty}^{\infty} f(x-t)\,dL(t). \tag{31}$$

We define two new functions by

$$\lambda(x)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(x-t)^2}\,dL(t) \tag{32}$$

and

$$h(x)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} g(x-t)\,e^{-t^2}\,dt. \tag{33}$$

Now (31), (32) and (33) imply that

$$h(x)=\int_{-\infty}^{\infty} f(x-u)\,\lambda(u)\,du. \tag{34}$$

Indeed, if we substitute (31) into (33) we find

$$h(x)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}\,dt\int_{-\infty}^{\infty} f(x-t-u)\,dL(u)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}\,dt\int_{-\infty}^{\infty} f(x-u)\,d_uL(u-t).$$

Since $f(t)$ is continuous and bounded and $L(u)$ of bounded variation it is easy to see that we may integrate first under the differential sign d_u obtaining

$$h(x)=\int_{-\infty}^{\infty} f(x-u)\,d_u\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} L(u-t)\,e^{-t^2}dt=\int_{-\infty}^{\infty} f(x-u)\,\frac{d}{du}\left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(u-t)^2}\,L(t)\,dt\right)du.$$

Now

$$\frac{d}{du}\left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(u-t)^2} L(t)\,dt\right) = -\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} 2(u-t)\,e^{-(u-t)^2} L(t)\,dt =$$

$$= -\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} L(t)\,d_t e^{-(u-t)^2} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(u-t)^2} dL(t) = \lambda(u)$$

and (34) is established.

By assumption we have $v(g) \leqq v(f)$; but also (33) is a variation-diminishing transformation by the direct part of Theorem 1, hence $v(h) \leqq v(g)$. It follows that $v(h) \leqq v(f)$ and that (34) is therefore a variation-diminishing transformation. Since $\lambda(x)$ is obviously continuous and summable we learn from the previous case already settled, that $\lambda(x)$ is a P. f. f. Therefore

$$\text{(35)} \qquad \int_{-\infty}^{\infty} e^{-sx}\lambda(x)\,dx = \frac{1}{\Psi(s)} = Ce^{\gamma s^2+\delta s}\prod_{\nu=1}^{\infty}\frac{e^{\delta_\nu s}}{1+\delta_\nu s} \qquad (-a < Rs < a),$$

where $C \gtrless 0$, $\gamma \geqq 0$, $0 < \gamma + \Sigma\delta_\nu^2 < \infty$.

Let us consider now the characteristic function of $L(t)$:

$$\text{(36)} \qquad H(it) = \int_{-\infty}^{\infty} e^{-itx}\,dL(x) \qquad (-\infty < t < \infty).$$

The convolution relation (32) now implies the characteristic function relation

$$\frac{1}{\Psi(it)} = e^{-\frac{t^2}{4}} H(it) \qquad (-\infty < t < \infty),$$

or

$$\text{(37)} \qquad H(it) = Ce^{\left(\frac{1}{4}-\gamma\right)t^2+i\delta t}\prod_{\nu=1}^{\infty}\frac{e^{i\delta_\nu t}}{1+i\delta_\nu t} \qquad (-\infty < t < \infty).$$

Passing to norms we have

$$|H(it)|^2 = C^2 e^{2\left(\frac{1}{4}-\gamma\right)t^2} \Big/ \prod_{\nu=1}^{\infty}(1+\delta_\nu^2 t^2),$$

and from this relation we can easily conclude that

$$\text{(38)} \qquad \frac{1}{4} - \gamma \leqq 0.$$

Indeed, let $\varepsilon > 0$ be given and choose n such that $\sum_{n+1}^{\infty}\delta_\nu^2 < \varepsilon$; since $|H(it)|^2$ is bounded, by (36), we have for t^2 sufficiently large the relations

$$C^2 e^{2\left(\frac{1}{4}-\gamma\right)t^2} = |H(it)|^2 \prod_{1}^{n}(1+\delta_\nu^2 t^2)\prod_{n+1}^{\infty}(1+\delta_\nu^2 t^2) < e^{\varepsilon t^2} e^{\varepsilon t^2} = e^{2\varepsilon t^2},$$

hence $\frac{1}{4} - \gamma < \varepsilon$ and thence (38), on letting $\varepsilon \to 0$. With $\gamma_1 = \gamma - \frac{1}{4} \geqq 0$

we may now rewrite (37) as

$$H(it)=Ce^{-\gamma_1 t^2+i\delta t}\prod_{\nu=1}^{\infty}\frac{e^{i\delta_\nu t}}{1+i\delta_\nu t}\qquad(-\infty<t<\infty). \tag{39}$$

Two cases may now arise:

1. $\gamma_1=0$ and $\delta_1=\delta_2=\ldots=0$. In this case (39) becomes

$$H(it)=\int_{-\infty}^{\infty}e^{-itx}dL(x)=Ce^{i\delta t},$$

hence

$$L(x)=\begin{cases}0 & \text{if}\quad x<-\delta\\ C & \text{if}\quad x>-\delta.\end{cases}$$

2. $\gamma_1+\sum_1^\infty \delta_\nu^2>0$, then $\Psi_1(s)=C^{-1}e^{-\gamma_1 s^2-\delta s}\prod_{\nu=1}^{\infty}(1+\delta_\nu s)\,e^{-\delta_\nu s}$

is of the type of (3); hence its reciprocal may be written as

$$\frac{1}{\Psi_1(s)}=\int_{-\infty}^{\infty}e^{-sx}\,\varepsilon\Lambda(x)\,dx\qquad(-a<Rs<a),$$

where $\varepsilon=\pm1$ and $\Lambda(x)$ is a P. f. f. From the last relation, (39) and (36) we now obtain the identity $\int_{-\infty}^{\infty}e^{-itx}\,dL(x)=\int_{-\infty}^{\infty}e^{-itx}\,\varepsilon\Lambda(x)\,dx\quad(-\infty<t<\infty)$,

from which we conclude that $L(x)=\varepsilon\int_{-\infty}^{x}\Lambda(u)\,du$.

This concludes the proof of our Theorem 1.

5. The connection of Theorem 1 with a theorem of Pólya.

In 1915 PÓLYA[6]) discovered the following

Theorem 3 (PÓLYA). *Let*

$$\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!}\mu_\nu s^\nu\qquad(\mu_0\neq0) \tag{40}$$

be a given formal power series with the following property (S): *If* $f(x)$ *is an arbitrary real polynomial, then the number of real zeros of the polynomial*

$$g(x)=\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!}\mu_\nu f^{(\nu)}(x) \tag{41}$$

should never exceed the number of real zeros of the given polynomial $f(x)$. *A power series* (40) *enjoys the property* (S) *if and only if it is the Taylor*

[6]) G. PÓLYA, Algebraische Untersuchungen über ganze Funktionen vom Geschlechte Null und Eins, *Journal für die reine und angewandte Math.*, **145** (1915), pp. 224–249, especially p. 231.

expansion of the reciprocal $1/\Psi(s)$ *of a function of the type* (3), *where, however, the possibility that* $\gamma=\delta_1=\delta_2=\ldots=0$ *is not excluded.*

This theorem is the source of the author's work on "Pólya frequency functions" and also explains the terminology. We now wish to point out that Theorem 3, combined with the representation (2), (3), for the Laplace transforms of Pólya frequency functions, will imply the converse part of our Theorem 1 without much difficulty, provided we make some additional assumption concerning the existence of the moments of $dL(t)$. The following assumption is perhaps stronger than necessary but applies easily: We add to Theorem 1 the additional assumption that

$$(42) \qquad f(s)=\int_{-\infty}^{\infty} e^{-st}dL(t) \qquad \text{converges in a strip } -a<Rs<a.$$

Indeed, (4) is now meaningful if $f(x)$ is a polynomial of degree n, say. Setting

$\mu_\nu=\int_{-\infty}^{\infty} t^\nu dL(t)$, we find that

$$g(x)=\int_{-\infty}^{\infty} f(x-t)\,dL(t)=\int_{-\infty}^{\infty}\sum\frac{(-t)^\nu}{\nu!}f^{(\nu)}(x)\,dx,$$

which may now be written in the form (41). The variation-diminishing property of (4), assumed for bounded functions, is readily shown to hold for polynomials. Once this done, we easily see that the transformation (41) also enjoys the property (S) required by Theorem 3. That $\mu_0\neq0$ is seen as follows: Consider $\lambda(x)$ as defined by (32). By assumption $v(\lambda)=0$. However, $\mu_0=0$ would clearly imply $\int_{-\infty}^{\infty}\lambda(x)\,dx=0$ which contradicts $v(\lambda)=0$ and the fact that $\lambda(x)\not\equiv0$ (because $L(t)\not\equiv0$). By Theorem 3 we now learn that

$$\int_{-\infty}^{\infty} e^{-st}dL(t)=\sum_0^\infty\frac{(-1)^\nu}{\nu!}\mu_\nu s^\nu=\frac{1}{\Psi(s)}.$$

If $\Psi(s)$ reduces to $Ce^{-\delta s}$, then $L(t)$ reduces to a step-function with only one jump. If $\gamma+\Sigma\delta_\nu^2>0$, then we have an identity

$$\int_{-\infty}^{\infty} e^{-st}dL(t)=\int_{-\infty}^{\infty} e^{-st}\varepsilon\Lambda(x)\,dx \qquad (-a<Rs<a),$$

where $\varepsilon=\pm1$ and $\Lambda(x)$ is a P. f. f. This concludes our proof.

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